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Analytical solution to a notch tip field in rubber-like materials under tension

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Abstract

The large strain field near a notch tip in some kinds of rubber materials under tension is reinvestigated. The completely analytical solution of the mapping functions are obtained for both expanding sector and shrinking sector. The stress singularity is expressed by notch angle precisely. This solution is derived from a typical elastic law but it is proved to be valid for a wide kind of rubber material.

1. Introduction

The stress singularity near a notch tip is an important problem for design when fracture is considered. Within the framework of linear elasticity, this problem has been studied by Williams (1952). For rubber-like materials that can endure very large strain, the analysis of singular field must be based on the nonlinear elasticity theory. When strain is really large, the K field and HRR field are no longer valid to represent the stress and strain states near a crack tip. Some effects of finite deformation on homogeneous and interfacial fracture were summarized by Geubelle (1995). The large strain fields near a notch tip were analyzed by Gao and Gao (1996) and Wang and Gao (1997) based on two different kinds of elastic laws proposed by Gao (1990, 1997), respectively. The solution for shrinking sector obtained by Gao and Gao (1996) and Wang and Gao (1997) are given numerically, the closed mathematical solution is obtained in the present paper so that the stress singularity is expressed by notch angle.

It is shown that the notch tip stress field obtained by both Gao and Gao (1996) and Wang and Gao (1997) are in uniaxial tension state. This feature is similar to that obtained by Knowles and Sternberg (1973) for crack that is based on a quite different elastic law. The interesting questions

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are (1) why the notch tip fields possess the same feature for various elastic laws? (2) what is the relation between the analysis given by Gao and Gao (1996) and Knowles and Sternberg (1973)?

2. Basic formulae

Consider a three-dimensional material domain. Let x^i ($i = 1, 2, 3$) denote the Lagrangian coordinate of a point. P and Q denote the position vectors of a point before and after deformation, respectively. Two local triads are defined as

$$P_i = \frac{\partial P}{\partial x^i}, \quad Q_i = \frac{\partial Q}{\partial x^i} \tag{1}$$

The displacement gradient tensor is given by

$$F = Q_i \otimes P^i \tag{2}$$

where the summation rule is implied, $\otimes$ is the dyadic symbol, P^i is the conjugate of P_i. The following three invariants of deformation will be used,

$$\begin{cases} I = (Q_i \cdot Q_j) \cdot (P^i \cdot P^j), \quad I_{-1} = (Q^i \cdot Q^j) \cdot (P_i \cdot P_j) \\ J = (Q_1, Q_2, Q_3)/(P_1, P_2, P_3) \end{cases} \tag{3}$$

in which $(*_1, *_2, *_3)$ denotes mixed product of $*_1$, $*_2$ and $*_3$.

Let U denote the strain energy per unit undeformed volume, the Cauchy stress can be given as

$$\tau = J^{-1} \frac{\partial U}{\partial Q_i} \otimes Q_i \tag{4}$$

The following relations are important,

$$\frac{\partial I}{\partial Q_i} \otimes Q_i = 2d, \quad \frac{\partial I_{-1}}{\partial Q_i} \otimes Q_i = -2d^{-1}, \quad \frac{\partial J}{\partial Q_i} \otimes Q_i = J \cdot E \tag{5}$$

where

$$d = F \cdot F^{\mathrm{T}}, \quad E = P_i \otimes P^i = Q_i \otimes Q^i \tag{6}$$

The equilibrium equation can be written as

$$Q^i \cdot \frac{\partial \tau}{\partial x^i} = 0 \tag{7}$$

Three kinds of strain energy used by Gao (1990, 1997) and Knowles et al. (1973) are

$$U_{G1} = a^*(I/J^{2/3})^n + b(J^2 - 1)^m J^{-2l} \tag{8}$$

$$U_{G2} = a(I^N + I_{-1}^N) \tag{9}$$

$$U_{KS} = \left(AI + BJ + C\frac{I}{J^2}\right)^N \tag{10}$$

in the present paper the energy (9) is used. The relation of (8)–(10) will be discussed later.

3. Expanding sector

Consider a two-dimensional notched specimen, before and after loading the specimen is shown in Fig. 1(a) and (b). Since the deformation near the notch tip is very large, as analyzed by Gao and Gao (1996), the deformation cannot be described by a uniform mapping function. Therefore, the field is divided into one expanding sector EX and two shrinking sectors, SH, SH' as shown in Fig. 1. Before loading, EX is very narrow while SH and SH' occupy almost the whole notch tip field. After loading, EX becomes very wide and occupies almost the whole notch tip field, but SH and SH' become very narrow.

Two Lagrangian coordinates are introduced (R, Θ, Z) are the cylindrical coordinates before deformation while (r, θ, z) are cylindrical coordinates after deformation. Under plane strain condition, the mapping functions from (R, Θ, Z) to (r, θ, z) for sector EX are assumed as

$$r = R^{1+\beta}\rho(\xi), \quad \xi = \Theta R^{-\alpha}, \quad -\infty < \xi < \infty$$
$$\theta = \omega(\xi), \quad z \equiv Z, \tag{11}$$

where α, β are positive constants to be determined. $|\Theta| < \Theta_0$, Θ_0 is a very small positive number. According to (1) and (11), the local triads are obtained,

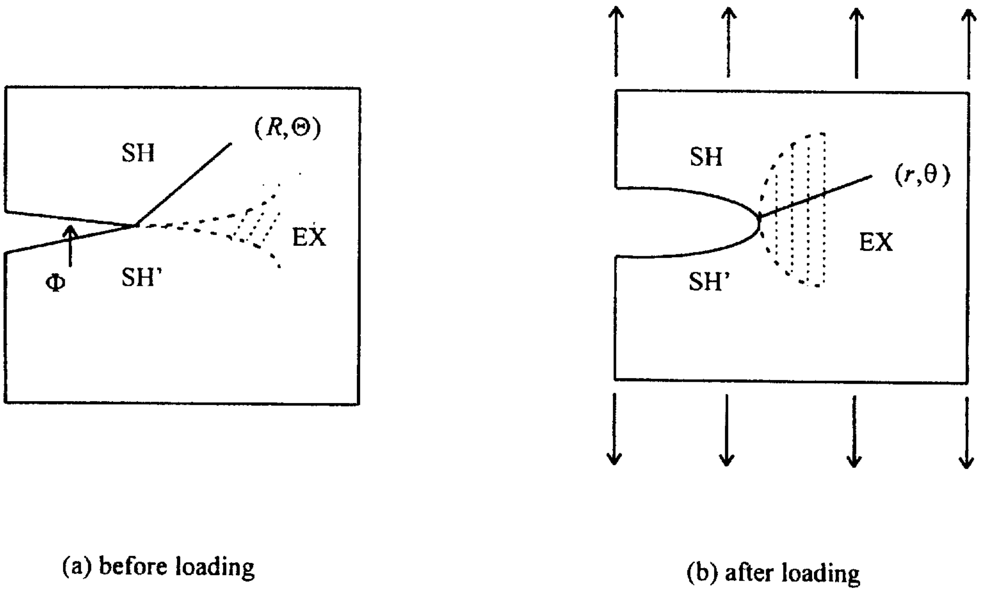

Fig. 1. Notched rubber specimen (a) before loading, (b) after loading.

$$\begin{cases} Q_R = R^{\beta}\{[(1+\beta)\rho-\alpha\xi\rho']e_r-\alpha\xi\rho\omega' e_\theta\} \\ Q_\Theta = R^{\beta-\alpha+1}(\rho' e_r-\rho\omega' e_\theta) \end{cases} \tag{12}$$

in which e_r, e_θ are the unit vectors, i.e.

$$e_r = Q_r = \frac{\partial Q}{\partial r}, \quad e_\theta = \frac{1}{r}Q_\theta = \frac{1}{r}\frac{\partial Q}{\partial \theta} \tag{13}$$

Noting that $P^R \cdot P^R = 1$, $P^R \cdot P^\Theta = 0$, $P^\Theta \cdot P^\Theta = R^{-2}$, using (3) and (12), neglecting the higher-order terms of R, the invariants are obtained,

$$I = R^{2\beta-2\alpha}u, \quad J = R^{2\beta-\alpha}v, \quad I_{-1} = R^{-2\beta}uv^{-2} \tag{14}$$

where

$$u = \rho'^2+\rho^2\omega'^2, \quad v = (1+\beta)\rho^2\omega' \tag{15}$$

In this paper the energy expression (9) is used, then according to (4)–(6) the Cauchy stress τ is

$$\tau = 2NaJ^{-1}(I^{N-1}d-I_{-1}^{N-1}d^{-1}) \tag{16}$$

where

$$\begin{aligned} d &= R^{2\beta-2\alpha}[\rho'^2 e_r\otimes e_r+\rho^2\omega'^2 e_\theta\otimes e_\theta+\rho'\rho\omega'(e_r\otimes e_\theta+e_\theta\otimes e_r)] \\ d^{-1} &= R^{-2\beta}v^{-2}[\rho'^2 e_\theta\otimes e_\theta+\rho^2\omega'^2 e_r\otimes e_r-\rho'\rho\omega'(e_r\otimes e_\theta+e_\theta\otimes e_r)] \end{aligned} \tag{17}$$

As analyzed in [3], for the case $\alpha = 2\beta$, the two terms in (16) are the same order, but finally there is no solution because it cannot match with the shrinking sectors. So we assume that $\alpha > 2\beta$, then only the first term in (16) is dominant, therefore,

$$\tau = R^{-\lambda}2Nav^{-1}u^{N-1}[\rho'^2 e_r\otimes e_r+\rho^2\omega'^2 e_\theta\otimes e_\theta+\rho'\rho\omega'(e_r\otimes e_\theta+e_\theta\otimes e_r)] \tag{18}$$

in which

$$\lambda = (2N-1)\alpha-2(N-1)\beta \tag{19}$$

Substituting (18) into (7), results in

$$\rho''-\rho\omega'^2 = 0, \quad \omega''+2\frac{\rho'}{\rho}\omega' = 0 \tag{20}$$

the symmetry solutions to eqn (20) are

$$\rho = \rho_0(1+k^2\xi^2)^{1/2}, \quad \omega = \operatorname{arc\,tg}(k\xi) \tag{21}$$

Where ρ_0 and k are constants to be determined by assembling different sectors.

Substituting (21) into (18) and noting (11), it follows,

$$\tau = R^{-\lambda}2Nav^{-1}u^{N-1}k^2\rho_0^2[\sin^2\theta e_r \otimes e_r + \cos^2\theta e_\theta \otimes e_\theta + \sin\theta\cos\theta(e_r \otimes e_\theta + e_\theta \otimes e_r)]$$

$$|\theta| < \frac{\pi}{2} \quad (22)$$

(22) means that in sector EX ($|\theta| < \pi/2$) the stress state is uniaxial tension.

4. Shrinking sectors

It can be seen from (21) that when $\xi \to \pm\infty$ function $\rho \to \infty$, so the mapping functions (11) become not valid, i.e. the problem must be considered in sectors SH (and SH'). In the shrinking sector SH, we assume that

$$r = R^{1-\delta}\varphi(\Theta), \quad \theta = \frac{\pi}{2} - R^{\gamma}\psi(\Theta), \quad z \equiv Z \quad (23)$$

where δ, γ are positive constants to be determined.

According to (1), (13) and (23) the local triads are obtained,

$$Q_R = R^{-\delta}\varphi[(1-\delta)e_r - \gamma R^{\gamma}\psi e_\theta]$$
$$Q_\Theta = R^{1-\delta}(\varphi' e_r - R^{\gamma}\varphi\psi' e_\theta) \quad (24)$$

From (3) and (24), the dominant terms of invariants are given as

$$I = R^{-2\delta}p, \quad J = R^{\gamma-2\delta}q, \quad I_{-1} = R^{2\delta-2\gamma}pq^{-2} \quad (25)$$

where

$$p = \varphi'^2 + (1-\delta)^2\varphi^2$$
$$q = \varphi[\gamma\varphi'\psi - (1-\delta)\varphi\psi'] \quad (26)$$

The Cauchy stress is also given by (16), but d and d^{-1} is given by

$$d = R^{-2\delta}[pe_r \otimes e_r - R^{\gamma}S(e_r \otimes e_\theta + e_\theta \otimes e_r) + R^{2\gamma}Te_\theta \otimes e_\theta] \quad (27)$$

$$d^{-1} = R^{2\delta-2\gamma}q^{-2}[pe_\theta \otimes e_\theta + R^{\gamma}S(e_r \otimes e_\theta + e_\theta \otimes e_r) + R^{2\gamma}Te_r \otimes e_r] \quad (28)$$

in which

$$S = \varphi[\gamma(1-\delta)\varphi\psi + \varphi'\psi'], \quad T = \varphi^2(\gamma^2\psi^2 + \psi'^2) \quad (29)$$

In order to derive the equilibrium equation, the following relations that are obtained from (23) are needed,

$$\frac{\partial}{\partial r} = R^{\delta}\varphi q^{-1}\left(-\psi'\frac{\partial}{\partial R} + \frac{\gamma\psi}{R}\frac{\partial}{\partial\Theta}\right), \quad \frac{1}{r}\frac{\partial}{\partial\theta} = R^{\delta-\gamma}q^{-1}\left(-\varphi'\frac{\partial}{\partial R} + \frac{1-\delta}{R}\varphi\frac{\partial}{\partial\Theta}\right) \quad (30)$$

eqn (30) implies that

$$\frac{1}{r}\frac{\partial}{\partial\theta} \sim R^{-\gamma}\frac{\partial}{\partial\gamma} \tag{31}$$

From (7) and (31) it follows that

$$\tau^{\theta\theta} \sim R^{\gamma}\cdot\tau^{r\theta} \sim R^{2\gamma}\cdot\tau^{rr} \tag{32}$$

According to (32), (16), (25), (27) and (28) the relation between γ and δ is matched,

$$\gamma = 2N\delta/(N+1) \tag{33}$$

then (16) is reduced to

$$\tau = R^{-\lambda}\cdot 2Nap^{N-1}q^{-1}[pe_r\otimes e_r - R^{\gamma}S(e_r\otimes e_\theta + e_\theta\otimes e_r) + R^{2\gamma}(T-pq^{-2N})e_\theta\otimes e_\theta] \tag{34}$$

in which

$$\gamma = 2\delta\left(N-\frac{1}{N+1}\right) \tag{35}$$

(34) shows that only the component τ^{rr} is dominant, on the other hand, the sector SH is located at the line of $\theta = \pi/2$, therefore, in sector SH the stress state is uniaxial tension along $\theta = \pi/2$. This is consistent with that obtained by Knowles and Sternberg (1973) for crack problems.

Equations (7), (30) and (32) can be used to get

$$\left(\lambda\psi'+\gamma\psi\frac{\mathrm{d}}{\mathrm{d}\Theta}+\frac{q}{\varphi^2}\right)(q^Nq^{-1})-\left[(\lambda-\gamma)\frac{\varphi'}{\varphi}+(1-\delta)\frac{\mathrm{d}}{\mathrm{d}\Theta}\right](p^{N-1}Sq^{-1})=0$$

$$\left[(\lambda-2\gamma)\frac{\varphi'}{\varphi}+(1-\delta)\frac{\mathrm{d}}{\mathrm{d}\Theta}\right](p^{N-1}Tq^{-1}-p^Nq^{-2N-1})-\left[(\lambda-\gamma)\psi'\right.$$

$$\left.+\gamma\psi\frac{\mathrm{d}}{\mathrm{d}\Theta}+\frac{2q}{\varphi^2}\right](p^{N-1}Sq^{-1})=0 \tag{36}$$

The first of (36) finally is reduced to

$$[\varphi''+(1-\delta)^2\varphi][1+2(N-1)\varphi'^2p^{-1}]-(1-\delta)(\lambda-\gamma)\varphi=0 \tag{37}$$

(37) is an equation that only contains the unknown φ.

The second of (36) can be reduced to

$$\psi''[1+(2N+1)(1-\delta)^2\varphi^2pq^{-2N-2}]+\varphi''\left[2(N-1)\frac{\varphi'\psi'}{p}-(1-\delta)^2\varphi\left(2NS+\frac{\gamma\psi p}{1-\delta}\right)q^{-2N-2}\right]$$

$$+\frac{2S}{\varphi^2}-\gamma(\lambda-2\gamma)\psi+\frac{2}{p}(1-\delta)^2(N-1)\varphi\varphi'\psi'+\frac{\varphi'}{\varphi}q^{-2N-2}[(1-\delta)(1-\delta-\gamma)(2N+1)p\varphi^2\psi'$$

$$-(1-\delta-\lambda+2\gamma)pq-2N(1-\delta)\varphi'^2q]=0 \tag{38}$$

5. Boundary conditions

For sector SH, the mapping functions φ, ψ must meet the following natural boundary conditions

$$\varphi(0) = 0 \tag{39}$$

$$\psi(0) = \infty \tag{40}$$

so that the displacements are continuous when sector SH is connected with sector EX.

Further, let Φ denote the notch angle, then at the notch margin $\Theta = \Theta^* = \pi - \Phi/2$, the traction free conditions must be satisfied, i.e.

$$\tau \cdot Q^{\Theta} = 0 \quad \text{at } \Theta = \Theta^* = \pi - \frac{\Phi}{2} \tag{41}$$

On the other hand, the expression (34) is given in coordinate r, θ, so it cannot be used directly for eqn (41). In order to consider conditions (41), the eqn (24) is inverted as

$$e_r = R^{\delta}\varphi\left(\frac{\gamma\psi}{R}Q_{\Theta} - \psi' Q_R\right)q^{-1}$$

$$e_{\theta} = R^{\delta-\gamma}\left(\frac{1-\delta}{R}\varphi Q_{\Theta} - \varphi' Q_R\right)q^{-1} \tag{42}$$

Using (42) and (34) we obtain

$$\tau \sim p^{N-1}\frac{R^{\delta-1}}{q^2}\{\gamma\varphi\psi(pe_r - R^{\gamma}Se_{\theta}) - (1-\delta)\varphi[Se_r - R^{\gamma}(T - pq^{-2N})e_{\theta}]\} \otimes Q_{\Theta} + (\cdots) \otimes Q_R \tag{43}$$

From (41) and (43) it follows that at $\Theta = \Theta^*$

$$\{\varphi' e_r - R^{\gamma}\varphi[\psi' + (1-\delta)pq^{-2N-1}]e_{\theta}\}_{\Theta^*} = 0 \tag{44}$$

(44) is equivalent to

$$\varphi'(\Theta^*) = 0 \tag{45}$$

$$\psi'(\Theta^*) = -(1-\delta)^{-\frac{N-1}{N+1}}[\varphi(\Theta^*)]^{\frac{-2N}{N+1}} \tag{46}$$

6. Solution of the eigenvalue problem

Equation (37) and boundary conditions (39) and (45) can be solved analytically. From (37), after some manipulating, we obtain

$$p^{1-\delta}\left[p-2(1-\delta)\frac{N-1}{2N-1}\varphi^2\right]^{\delta} = \text{const} \tag{47}$$

Let

$$\varphi = \varsigma \sin\eta, \quad \varphi' = (1-\delta)\varsigma\cos\eta \tag{48}$$

then (47) gives

$$\varsigma = \varsigma_0[N-(2N-1)\delta+(N-1)\cos 2\eta]^{-\frac{\delta}{2}} \tag{49}$$

in which ς_0 is a constant.

Using (48) and (49) φ can be eliminated, then

$$\frac{d\eta}{d\Theta}\left[1+\frac{(2N-1)\delta}{N-(2N-1)\delta+(N-1)\cos 2\eta}\right] = 1 \tag{50}$$

Noting (48) and boundary condition (39), the particular solution of (50) is

$$\Theta = \eta + \frac{\delta}{\varepsilon(1-\delta)}\operatorname{arc\,tg}(\varepsilon\,\operatorname{tg}\eta) \tag{51}$$

where

$$\varepsilon = \left[\frac{1-(2N-1)\delta}{(2N-1)(1-\delta)}\right]^{1/2} \tag{52}$$

At the notch margin, boundary condition (45) requires that $\eta = \pi/2$, then (51) gives

$$\frac{\pi}{2}\left[1+\frac{\delta}{\varepsilon(1-\delta)}\right] = \Theta^* = \pi - \frac{\Phi}{2} \tag{53}$$

eqn (53) shows that only when $\Phi < \pi$, the notch tip possesses singularity ($\delta > 0$). From (53), δ can be expressed precisely,

$$\delta = \frac{Ne}{(2N-1)(1-e)}\left[\sqrt{1+\frac{1-e}{N^2e}(2N-1)}-1\right] \tag{54}$$

in which

$$e = \left(1-\frac{\Phi}{\pi}\right)^2. \tag{55}$$

By means of (48), (49) and (51), the solution of (37) is given with a free parameter ς_0 that indicates the amplitudes of the field.

Using (48), (49) we have

$$\varphi(\Theta^*) = [1-(2N-1)\delta]^{-\frac{\delta}{2}}\varsigma_0$$

$$\varphi'(0) = (1-\delta)^{1-\frac{\delta}{2}}(2N-1)^{-\frac{\delta}{2}}\varsigma_0 \tag{56}$$

The extreme case is $\Phi = 0$, then the notch becomes a crack, and (53) becomes $\varepsilon = \delta/(1-\delta)$, $\delta = 1/(2N)$. By inverting (51) we have

$$\operatorname{tg}\eta = \frac{N}{\sin\Theta}(\Omega-\cos\Theta) \tag{57}$$

where

$$\Omega = \left[1-\left(1-\frac{1}{N}\right)^2 \sin^2\Theta\right]^{1/2} \tag{58}$$

From (48), (49) and (57), it follows

$$\varphi = \varsigma_0\left(\frac{N}{2}\right)^{\frac{1-\delta}{2}}(\Omega-\cos\Theta)^{1/2}\left[\Omega+\left(1-\frac{1}{N}\right)\cos\Theta\right]^{\frac{1}{2}-\frac{1}{2N}} \tag{59}$$

The expression (59) is essentially the same as the solution given by Knowles and Sternberg (1973) for a crack.

7. The solution of $\psi(\Theta)$

When φ is given, eqn (38) with boundary conditions (40) and (46) can be solved numerically as performed by Wang and Gao (1997). The boundary condition (40) is always satisfied automatically but the boundary value $\psi(\Theta^*)$ is a free parameter that is expected to be determined by the condition given in Section 10. The analysis of (38) shows that when $\Theta \to 0$,

$$\psi(\Theta) \cong C_\psi \Theta^{-1}, \quad \Theta \to 0 \tag{60}$$

where C_ψ is a constant dependent on the values of both ς_0 and $\psi(\Theta^*)$.

8. Comparison of three elastic laws

The elastic law (8) contains two terms, i.e. the deviator and the hydrostatical stress. Elastic law (9) also contains two terms which reflect the response to tension and compression, respectively. In elastic law (10), the first term represents the response to tension while the second term is hydrostatic stress. Certainly, above three of elastic law possess different physical meaning. However, the solution to crack (or notch) tip field obtained by Gao and Gao (1996), Wang and Gao (1997) and Knowles and Sternberg (1973) appears some common features, i.e. all of the fields are in uniaxial tension state. The solutions given by Gao and Gao (1996) and Wang and Gao (1997) for expanding sector are the same, in shrinking sector are also the same for function φ. In the solution given by Knowles and Sternberg (1973), the expanding sector was ignored but the solution is essentially the same with that given by Gao and Gao (1996) and Wang and Gao (1997) in shrinking sector ($\varphi \leftrightarrow v_2, \varphi\psi \leftrightarrow v_1$).

In order to explain above interesting facts we should write down the stresses according to strain energies given by Gao (1990, 1997) and Knowles and Sternberg (1973),

$$\tau_{G1} = 2na^* I^{n-1} J^{-\frac{2n}{3}-1}\left(d - \frac{I}{3}E\right) + 2b(J^2-1)^{m-t} J^{-2t-1}[(m-t)J^2+t]E$$

$$\tau_{G2} = 2NaJ^{-1}(I^{N-1}d - I_{-1}^{N-1}d^{-1})$$

$$\tau_{KS} = 2NJ^{-1}(AI+BJ+CIJ^{-2})^{N-1}\left[(A+CJ^{-2})d + J\left(\frac{B}{2} - CIJ^{-3}\right)E\right] \tag{61}$$

where d and d^{-1} are given by (17), (27) and (28), respectively, for different sectors.

The analysis by Gao and Gao (1996), Wang and Gao (1997) and Knowles and Sternberg (1973) shown that the dominant terms of stress near the crack (or notch) tip are reduced to

$$\tau_{G1} = 2na^* J^{-1} I^{n-1} J^{-\frac{2n}{3}} d, \quad J = \left[\frac{na^* I^n}{3(m-t)b}\right]^{\frac{3}{2n+6(m-t)}} \tag{62}$$

$$\tau_{G2} = 2NaJ^{-1} I^{N-1} d \tag{63}$$

$$\tau_{KS} = 2NA^N J^{-1} I^{N-1} d, \quad J = \left(\frac{2C}{B} I\right)^{1/3} \tag{64}$$

Comparing the eqns (62)–(64) we can see that the three elastic laws possess the same essence provided,

$$\frac{3(m-t)n}{3(m-t)+n} = N, \quad A = a^{1/N}, \quad \frac{na^*}{N}\left[\frac{na^*}{3(m-t)b}\right]^{\frac{-n}{n+3(m-t)}} = a \tag{65}$$

that is why all of the three elastic laws give the same solutions for expanding sector and the same solution of φ for shrinking sector. The difference of above three elastic laws only influence the solution of function ψ in shrinking sectors.

It should be noted that the value of δ obtained by Gao (1990, 1997) and Knowles and Sternberg (1973) are given as $\delta_{G1} = 1/(2n) + [1/(6(m-t))]$, $\delta_{G2} = 1/(2N) = \delta_{KS}$ this is consistent with the first of (65).

Next, consider the solution of ψ in sector SH for elastic laws (62) and (64).

According to the second of (62), it follows that

$$(1-\delta)\psi' - \frac{\gamma}{\varphi}\varphi'\psi + \frac{1}{\varphi^2} P^{\frac{3n}{2n+6(m-t)}} \cdot \left[\frac{na^*}{3(m-t)b}\right]^{\frac{3}{2n+6(m-t)}} = 0 \tag{66}$$

eqn (66) can be solved numerically as performed by Gao and Gao (1996). The natural boundary condition $\psi(0) = \infty$ is satisfied automatically, but the initial value $\psi(\Theta^*)$ is a free parameter that cannot be determined by the asymptotic solution. When $\Theta \to 0$, it follows that,

$$\psi(\Theta) = C_\psi \Theta^{-1} \tag{67}$$

eqn (66) and the second of (56) give,

$$C_\psi = \frac{1}{1+\gamma-\delta}\left[\frac{na^*}{3(m-t)b}\right]^{\frac{3}{2n+6(m-t)}}[(1-\delta)^{1-\frac{\delta}{2}}(2N-1)^{-\frac{\delta}{2}}\varsigma_0]^{\frac{n-6(m-t)}{n+3(m-t)}} \tag{68}$$

where N is given by the first of (65).

According to the second of (64), it follows that

$$(1-\delta)\psi' - \frac{\gamma}{\varphi}\varphi'\psi + \left(\frac{2C}{B}\right)^{1/3}\frac{P^{1/3}}{\varphi^2} = 0 \tag{69}$$

eqn (69) can be solved numerically, but the initial value $\psi(\Theta^*)$ is a free parameter that cannot be determined by the asymptotic solution.

When $\Theta \to 0$, the asymptotic behavior of $\psi(\Theta)$ is also given by (67).

From the second of (56) and (69), it follows that,

$$C_\psi = \frac{1}{1+\gamma-\delta}\left(\frac{2C}{B}\right)^{1/3}[(1-\delta)^{1-\frac{\delta}{2}}(2N-1)^{-\frac{\delta}{2}}\varsigma_0]^{\frac{2N}{3}-2} \tag{70}$$

So, both for elastic laws (62) and (64), C_ψ is expressed by ς_0.

9. Assembly of sectors

The method of sector division does not mean that the different sectors are isolated by some strict boundaries. Actually, in sector EX when $\xi \to \infty$, the solution must be translated to that in sector SH when $\Theta \to 0$. Now, comparing the limiting expression of ρ, ω ($\xi \to \infty$) and φ, ψ ($\Theta \to 0$), we assemble the sectors EX and SH. When $\xi \to \infty$, from (21) we have

$$\rho = \rho_0 k\xi, \quad \omega = \frac{\pi}{2} - (k\xi)^{-1} \tag{71}$$

Substituting (71) into (11) it follows that

$$r = R^{1+\beta-\alpha}\rho_0 k\Theta, \quad \theta = \frac{\pi}{2} - R^\alpha (k\Theta)^{-1} \tag{72}$$

On the other hand, from the second of (56), (60), (67) and (23) we have,

$$r = R^{1-\delta}(1-\delta)^{1-\frac{\delta}{2}}(2N-1)^{-\frac{\delta}{2}}\varsigma_0\Theta$$

$$\theta = \frac{\pi}{2} - R^{\gamma}C_{\psi}\Theta^{-1} \tag{73}$$

Comparing (72) and (73), it is required that

$$\beta = \gamma - \delta, \quad \alpha = \gamma, \quad k = 1/C_{\psi}$$

$$\rho_0 = C_{\psi}(1-\delta)^{1-\frac{\delta}{2}}(2n-1)^{-\frac{\delta}{2}}\varsigma_0 \tag{74}$$

Thus, the mapping functions ρ, ω, φ and ψ are identical at the boundary of different sectors, so the displacements and stresses are continuous from one sector to another.

10. Additional requirement

For all the three elastic laws (8)–(10), the parameters ρ_0 and k of sector EX are related with ς_0 and C_{ψ} of sector SH by eqn (74). For elastic law (8) and (10), C_{ψ} is not an independent parameter because of (68) and (70). For elastic law (9), the value of C_{ψ} depends on both ς_0 and $\psi(\Theta^*)$. Since there is no restriction on $\psi(\Theta^*)$, C_{ψ} seems to be free.

If the ignored term d^{-1} in eqn (16) is considered, and we further require that $\tau^{rr} = 0$ at $\xi = 0$, then we have

$$I^{N-1}R^{2\beta-2\alpha}[\rho'^2 + R^{2\alpha}(1+\beta)^2\rho^2] = I_{-1}^{N-1}R^{-2\beta}v^{-2}\rho^2\omega'^2 \tag{75}$$

Since $\alpha = 2N\beta/(N-1)$, using (21) at $\xi = 0$ it follows that

$$k = (1+\beta)^{\frac{N+1}{N-1}}\rho_0^{-\frac{2N}{N-1}} \tag{76}$$

Further using relations (74), it follows that

$$C_{\psi} = \frac{1}{1+\beta}[\varsigma_0^{-1}(1-\delta)^{\frac{\delta}{2}-1}(2N-1)^{\frac{\delta}{2}}]^{\frac{2N}{N+1}} \tag{77}$$

The relation (77) provides a condition to determine the value of $\psi(\Theta^*)$ by the numerical solution of (38). Therefore, if the additional requirement $\tau^{rr} = 0$ (at $\xi = 0$) is reasonable, the field obtained for elastic law (63) only contains one parameter ς_0. Detailed analysis shows that the condition $\tau^{rr} = 0$ (at $\xi = 0$) means the stress state in expansion sector is exactly in uniaxial tension. It should be noted that for elastic laws (62) and (64), the initial value $\psi(\Theta^*)$ cannot be determined.

11. Concluding remarks

- The notch tip field of the rubber material discussed in this paper contains an expanding sector and two shrinking sectors.
- The solution given by Knowles and Sternberg (1973, 1974) is only for shrinking sectors that

become very narrow after deformation. The expanding sector that occupies almost the whole deformed crack tip field was ignored by them.

- The singularity of notch tip field is expressed by N and the notch angle Φ in eqn (54).
- The solution obtained by Gao and Gao (1996), Gao (1997) and Knowles and Sternberg (1973) possess the same feature, i.e. stress state is uniaxial tension. This fact can be explained by the common character of the elastic laws under certain strain state.
- The difference of elastic laws (8)–(10) only appears in the solution of ψ in sector SH. For elastic laws (8) and (10), $\psi(\Theta^*)$ cannot be determined by the asymptotic solution. For elastic law (9) with an additional condition, $\psi(\Theta^*)$ can be determined numerically.
- An interesting problem that is worthy to mention is that the common used K field ($\varepsilon \sim r^{-1/2}, \sigma \sim r^{-1/2}$) and HRR field ($\varepsilon \sim r^{-n/(n+1)}, \sigma \sim r^{-1/(n+1)}$) are only valid for small strain case ($\varepsilon \ll 1$). The solution of this paper is only for large strain case ($\varepsilon \gg 1$). There exists a finite strain solution ($\varepsilon \sim 1$) in between the K or HRR solutions and the solution for rubber. Therefore the solution of this paper cannot be directly connected with K or HRR fields.

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